JOURNAL OF COMPUTATIONAL PHYSICS 75, 73–85 (1988)

Initial and Boundary Conditions for Flux-Limited Diffusion Theory

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Received December 12, 1986; revised April 30, 1987

Initial and boundary layer analyses are performed to obtain the asymptotically correct initial and boundary conditions for the Levermore flux-limited diffusion approximation to the equation of radiative transfer. © 1988 Academic Press, Inc.

1. INTRODUCTION

In an article by Levermore and Pomraning [1], a flux-limited diffusion equation was derived as an approximation to the equation of radiative transfer. This diffusion equation is qualitatively more accurate than the classic diffusion approximations, namely isotropic (Eddington) diffusion theory [2, 3] and asymptotic diffusion theory [2, 4], in that it is flux-limited. By the term "flux-limited" we mean that the magnitude of the radiative flux cannot exceed the product of the radiation energy density and the speed of light. This flux-limited description has been incorporated into several radiative transfer codes [5] and for essentially infinite medium problems (those for which boundary effects are small) has been found to be quantitatively as well as qualitatively more accurate than the classic, non-flux-limited diffusion theories.

One of the shortcomings of the treatment of Levermore and Pomraning [1] is their handling of the initial and boundary conditions for the diffusion equation. Very ad hoc conditions were suggested in their paper, and the authors acknowledged that these conditions, in particular the boundary condition, needed to be reformulated in a better manner. An attempt to improve upon the boundary condition has recently been reported by Pomraning [6]. His analysis is a significant improvement over the original suggestion of Levermore and Pomraning [1] and leads to a boundary condition which predicts the exact interior solution for transfer problems when a single asymptotic mode is present. It is only for this class of problems for which the flux-limited diffusion equation of Levermore [7] and Levermore and Pomraning [1] has the capability of being exact, and with the Pomraning [6] boundary condition this capability is fulfilled. However, as pointed

0021-9991/88 $3.00

out in the concluding remarks of the Pomraning [6] paper, his boundary condition also has a somewhat ad hoc flavor to it. He suggests that his analysis, while a clear improvement over that of Levermore and Pomraning [1], be considered as an interim result, awaiting a rigorous boundary layer analysis which would give the boundary condition for the flux-limited diffusion equation in a completely non-ad hoc manner.

In this paper, we present that boundary (and initial) layer analysis and derive what we believe are the proper initial and boundary conditions for the flux-limited diffusion equation of Levermore [7] and Levermore and Pomraning [1]. The initial condition we obtain agrees with that suggested in an ad hoc manner by Levermore and Pomraning [1]. The boundary condition we derive contains the boundary condition of Pomraning [6] as a limiting case. The analysis we use closely parallels that of Larsen and Keller [3], who discussed the use of asymptotics to derive the isotropic (Eddington) diffusion approximation to the linear transport equation.

2. Formulation

The transport equation for the specific intensity of radiation, $\tilde{I}(\mathbf{r}, \boldsymbol{\Omega}, t)$, has the form [1, 2]

$$\frac{1}{c}\frac{\partial \tilde{I}}{\partial t} + \boldsymbol{\Omega} \cdot \nabla \tilde{I} + \sigma \tilde{I} = \frac{c}{4\pi}(\sigma_a B + \sigma_s \tilde{E}), \tag{1}$$

where $\mathbf{r}$, $\boldsymbol{\Omega}$, and t are the spatial, angular, and temporal variables; c is the speed of light; $\sigma_a(\mathbf{r}, t)$ is the absorption coefficient, suitably corrected for induced emission; $\sigma_s(\mathbf{r}, t)$ is the scattering coefficient; $\sigma = \sigma_a + \sigma_s$ is the total interaction coefficient; $B(\mathbf{r}, t)$ is the blackbody energy density; and $\tilde{E}(\mathbf{r}, t)$ is the radiation energy density as defined by

$$\tilde{E}(\mathbf{r}, t) = \frac{1}{c}\int_{4\pi} d\boldsymbol{\Omega}\, \tilde{I}(\mathbf{r}, \boldsymbol{\Omega}, t). \tag{2}$$

In writing Eq. (1) we have assumed local thermodynamic equilibrium for the matter and isotropic and coherent scattering of photons. Equations (1) and (2) may be considered to be either frequency dependent with the frequency variable suppressed, or grey (frequency integrated) equations. The initial and boundary conditions on Eq. (1) are given by

$$\tilde{I}(\mathbf{r}, \boldsymbol{\Omega}, 0) = \Lambda(\mathbf{r}, \boldsymbol{\Omega}), \tag{3}$$

$$\tilde{I}(\mathbf{r}_s, \boldsymbol{\Omega}, t) = \Gamma(\mathbf{r}_s, \boldsymbol{\Omega}, t), \qquad \mathbf{n} \cdot \boldsymbol{\Omega} < 0, \tag{4}$$

where $\mathbf{n}$ is a unit outward normal vector at the surface point $\mathbf{r}_s$, and the functions Λ and Γ are the prescribed initial and boundary data.

To use asymptotics to obtain a diffusion approximation to Eqs. (1) through (4), we assume that σ_a, σ_s, and B are slowly varying functions of their arguments $\mathbf{r}$ and t. Additionally, we assume that the function Λ in the initial condition given by Eq. (3) is slowly varying with respect to $\mathbf{r}$, and that the function Γ in the boundary condition given by Eq. (4) is slowly varying with respect to both $\mathbf{r}_s$ and t. Then, outside of any initial (early time) and boundary (near the surface) layers, it is reasonable to expect that $\tilde{I}$ will satisfy, to a good approximation, some simpler description of radiative transfer than that given by Eqs. (1) through (4). We obtain this simpler description, which will consist of a flux-limited diffusion equation and associated initial and boundary conditions for the energy density $\tilde{E}(\mathbf{r}, t)$, by asymptotics.

Following Larsen and Keller [3], we decompose the specific intensity of radiation $\tilde{I}(\mathbf{r}, \mathbf{\Omega}, t)$ into the sum of four intensities, namely: (1) $I(\mathbf{r}, \mathbf{\Omega}, t)$, the interior solution, presumed to be an accurate approximation to $\tilde{I}$ away from initial and boundary layers; (2) $I_{\mathrm{i}}(\mathbf{r}, \mathbf{\Omega}, t)$, the contribution to the initial (early time) layer (away from the boundary layer) which is presumed to decay rapidly in time; (3) $I_{\mathrm{b}}(\mathbf{r}, \mathbf{\Omega}, t)$, the contribution to the boundary (near surface) layer (away from the initial layer) which is presumed to decay rapidly in space in a direction normal to the surface; and (4) $I_{\mathrm{ib}}(\mathbf{r}, \mathbf{\Omega}, t)$, the contribution to the initial-boundary (early time and near surface) layer which is presumed to decay rapidly in both space and time. These four intensities satisfy the equations

$$\frac{1}{c}\frac{\partial I}{\partial t} + \mathbf{\Omega}\cdot\nabla I + \sigma I = \frac{c}{4\pi}(\sigma_a B + \sigma_s E), \tag{5}$$

$$\frac{1}{c}\frac{\partial I_x}{\partial t} + \mathbf{\Omega}\cdot\nabla I_x + \sigma I_x = \frac{c}{4\pi}\sigma_s E_x, \qquad x = \mathrm{i, b, ib}, \tag{6}$$

where

$$E = \frac{1}{c}\int_{4\pi} d\mathbf{\Omega}\, I(\mathbf{\Omega}), \qquad E_x = \frac{1}{c}\int_{4\pi} d\mathbf{\Omega}\, I_x(\mathbf{\Omega}), \tag{7}$$

with the total intensity given by

$$\tilde{I} = I + I_{\mathrm{i}} + I_{\mathrm{b}} + I_{\mathrm{ib}}. \tag{8}$$

Following Levermore and Pomraning [1], we analyze Eq. (5) by assuming that away from initial and boundary layers, the transport solution is separable in space-time and angle. This implies that we can write

$$I(\mathbf{r}, \mathbf{\Omega}, t) = cE(\mathbf{r}, t)\,\psi(\mathbf{\Omega}), \tag{9}$$

where, for consistency, we must have

$$\int_{4\pi} d\mathbf{\Omega}\,\psi(\mathbf{\Omega}) = 1. \tag{10}$$

The ansatz given by Eq. (9) is the basis of the flux-limited diffusion equation discussed by Levermore and Pomraning [1]. An earlier derivation of this diffusion equation has been given by Levermore [7], who used the Chapman–Enskog formalism from the kinetic theory of gases. By either treatment, the results are

$$\psi(\mathbf{\Omega}) = \frac{1}{4\pi}\left(\frac{1}{R\coth R - \mathbf{\Omega}\cdot\mathbf{R}}\right), \tag{11}$$

where the flux-limiting parameter R is given by

$$R = |\mathbf{R}|; \qquad \mathbf{R} = -\frac{\nabla E}{\sigma\omega E}, \tag{12}$$

and the effective albedo ω is defined by

$$\omega = \frac{\sigma_a B + \sigma_s E}{\sigma E}. \tag{13}$$

In this approximation, the radiation energy density $E(\mathbf{r}, t)$ satisfies the diffusion equation

$$\frac{1}{c}\frac{\partial E}{\partial t} - \nabla\cdot\left(\frac{\lambda}{\sigma\omega}\nabla E\right) = \sigma_a(B - E), \tag{14}$$

where λ is an explicit function of R given by

$$\lambda(R) = \frac{1}{R}\left(\coth R - \frac{1}{R}\right). \tag{15}$$

As discussed by Levermore and Pomraning [1], this diffusion equation is fully flux-limited, and contains both isotropic (Eddington) and asymptotic diffusion as limiting cases.

The required initial condition on Eq. (14) is derived by analyzing the initial layer problem for $I_i(\mathbf{r}, \mathbf{\Omega}, t)$, which we take up in the next section. Similarly, the boundary condition on Eq. (14) is obtained by analyzing the boundary layer problem for $I_b(\mathbf{r}, \mathbf{\Omega}, t)$, which is treated in Section 4 of this paper. The initial-boundary layer problem for $I_{ib}(\mathbf{r}, \mathbf{\Omega}, t)$ does not need to be considered unless one is specifically interested in the short time behavior of the solution near the boundary. In particular, an analysis of I_{ib} plays no role in obtaining the initial and boundary conditions on the flux-limited diffusion equation given by Eq. (14).

3. The Initial Layer

To obtain the initial condition on Eq. (14), we need to analyze the initial layer problem given by Eq. (6) with $x = \mathrm{i}$. If we neglect the spatial derivative in this equation, we have

$$\frac{1}{c}\frac{\partial I_i}{\partial t} + \sigma I_i = \frac{c}{4\pi}\sigma_s E_i. \tag{16}$$

The spatial derivative can be neglected since in the initial layer the time dependence is assumed strong, whereas the spatial dependence (away from boundary layers) is assumed weak. To be more quantitative, we note that outside of the boundary layer, Eq. (8) simplifies to, since the boundary layer contributions are essentially zero,

$$\tilde{I} = I + I_i, \tag{17}$$

and we assume that I_i decays exponentially in time with a characteristic time given by a mean collision time, or less. That is, I_i is assumed to fall off in time as $\exp(-c\sigma t)$, or faster. Any weaker time dependence is assumed to be carried by the interior solution $I(\mathbf{r}, \mathbf{\Omega}, t)$ in Eq. (17).

To solve Eq. (16), we note that this equation holds at each (interior) space point $\mathbf{r}$; $\mathbf{r}$ is simply a parameter in this equation. Further, the cross sections σ_s and σ in Eq. (16) are essentially independent of time and can be taken as their $t = 0$ values. Thus, Eq. (16) is a first-order linear equation with constant coefficients and hence easily solved. Integration of Eq. (16) over all solid angle gives

$$\frac{1}{c}\frac{\partial E_i}{\partial t} + \sigma_a E_i = 0, \tag{18}$$

which has the solution

$$E_i(t) = E_i(t=0)\, e^{-c\sigma_a t}, \tag{19}$$

where $E_i(t=0)$ is the constant of integration. Since $\sigma_a < \sigma$, Eq. (19) shows that $E_i(t)$ falls off too slowly to qualify as the fast scale initial layer contribution; we have already stated that $E_i(t)$ must fall off at least as fast as $\exp(-c\sigma t)$. Hence we must set $E_i(t=0) = 0$ to achieve

$$E_i(t) = 0, \tag{20}$$

which is the only acceptable solution to Eq. (16).

Now, using Eq. (17) in the initial condition given by Eq. (3) gives

$$I(\mathbf{r}, \mathbf{\Omega}, 0) + I_i(\mathbf{r}, \mathbf{\Omega}, 0) = A(\mathbf{r}, \mathbf{\Omega}). \tag{21}$$

Integration of Eq. (21) over all solid angle, making use of Eq. (20), gives

$$E(\mathbf{r}, 0) = \frac{1}{c} \int_{4\pi} d\mathbf{\Omega}\, A(\mathbf{r}, \mathbf{\Omega}), \tag{22}$$

as the initial condition for the flux-limited diffusion equation. This is the same initial condition suggested earlier on an ad hoc basis [1].

We note that Eq. (20) implies that there is no fast variation in the initial layer. On physical grounds, this is a very desirable result in that all of the initial energy in the problem is included in the initial condition for the diffusion equation. This implies energy conservation; the initial energy for the transport problem is all accounted for in the diffusion process.

4. The Boundary Layer

We now analyze the boundary layer equation for I_b, given by Eq. (6) with $x = \mathrm{b}$. This will yield the boundary condition for the diffusion equation. We neglect the time and tangential spatial derivatives in this equation because of the assumed dominance of the normal spatial derivative. We then have

$$\mu \frac{\partial I_b}{\partial z} + \sigma I_b = \frac{c}{4\pi} \sigma_s E_b, \tag{23}$$

where z is a local spatial coordinate normal to the surface at $\mathbf{r}_s$, and pointing into the medium. The variable μ is the cosine of the angle between the photon flight direction $\mathbf{\Omega}$ and the z axis; i.e., $\mu = -\mathbf{n} \cdot \mathbf{\Omega}$, where $\mathbf{n}$ is a unit outward normal vector at $\mathbf{r}_s$. Since I_b is assumed to decay rapidly with z, in particular faster than exponential with a mean free path characteristic distance (i.e., faster than $\exp(-\sigma z)$), Eq. (23) can be taken to hold for $0 \leqslant z < \infty$. Equation (23) holds at each (interior) time t; t is simply a parameter in Eq. (23). Further, the cross sections σ_s and σ in Eq. (23) are taken as their values at $\mathbf{r} = \mathbf{r}_s$. The boundary condition on Eq. (23) is found by combining Eqs. (4) and (8). One finds

$$I(\mathbf{r}_s, \mathbf{\Omega}, t) + I_b(0, \mathbf{\Omega}, t) = \Gamma(\mathbf{r}_s, \mathbf{\Omega}, t), \qquad \mu > 0, \tag{24}$$

where the zero argument in $I_b(0, \mathbf{\Omega}, t)$ means $z = 0$. In writing Eq. (24) we have neglected I_i and I_{ib}, since away from the initial layer these terms are negligibly small.

We remove the azimuthal (ϕ) dependence from Eqs. (23) and (24) by integrating over ϕ. If, for any function $h(\mathbf{\Omega})$, we define

$$\bar{h}(\mu) \equiv \int_0^{2\pi} d\phi\, h(\mathbf{\Omega}), \tag{25}$$

then an integration of Eqs. (23) and (24) over ϕ yields

$$\mu \frac{\partial \bar{I}_b(z, \mu, t)}{\partial z} + \sigma \bar{I}_b(z, \mu, t) = \frac{\sigma_s}{2} \int_{-1}^{1} d\mu' \, \bar{I}_b(z, \mu', t), \qquad 0 \leqslant z < \infty, \tag{26}$$

$$\bar{I}_b(0, \mu, t) = \bar{\Gamma}(\mathbf{r}_s, \mu, t) - \bar{I}(\mathbf{r}_s, \mu, t), \qquad \mu > 0. \tag{27}$$

Since $\bar{I}_b$ vanishes at $z = \infty$, we recognize Eqs. (26) and (27) as the classical halfspace albedo problem, with constant cross sections and with time t appearing simply as a parameter. The right-hand side of Eq. (27) plays the role of the incoming intensity for the $\bar{I}_b$ problem.

The solution of Eqs. (26) and (27) for $\bar{I}_b(z, \mu, t)$ can be represented as a linear combination of the Case eigenmodes [8], consisting of a single discrete decaying mode and the continuum decaying modes. For this solution to be a fast scale boundary layer, decaying faster than $\exp(-\sigma z)$, the discrete mode must not be present; it decays too slowly. This discrete mode is accounted for by the diffusion component (the interior solution I) in Eq. (8). Thus, as discussed by Case and Zweifel [8], the right-hand side of Eq. (27) must be orthogonal over $0 \leqslant \mu \leqslant 1$, with weight function $W(\mu)$, to the discrete mode. That is, we must have

$$\int_0^1 d\mu \, W(\mu) \, \phi_{o+}(\mu) [\bar{\Gamma}(\mathbf{r}_s, \mu, t) - \bar{I}(\mathbf{r}_s, \mu, t)] = 0. \tag{28}$$

Here $\phi_{o+}(\mu)$ is the discrete Case decaying mode which is given by [8]

$$\phi_{0+}(\mu) = \frac{\nu_o w}{2} \left(\frac{1}{\nu_o - \mu} \right), \tag{29}$$

where ν_o is the positive root of the dispersion relationship

$$1 = w \nu_o \tanh^{-1}(1/\nu_o), \tag{30}$$

and w is the single scatter albedo (at space point $\mathbf{r}_s$ and time t) given by

$$w = \sigma_s / \sigma. \tag{31}$$

The weight function $W(\mu)$, which also is a function of w, is discussed by Case and Zweifel [8]. Now, from Eq. (9) we have

$$\bar{I}(\mathbf{r}_s, \mu, t) = cE(\mathbf{r}_s, t) \, \bar{\psi}(\mu), \tag{32}$$

and hence Eq. (28) can be written

$$\int_0^1 d\mu \, W(\mu) \, \phi_{o+}(\mu) [\bar{\Gamma}(\mathbf{r}_s, \mu, t) - cE(\mathbf{r}_s, t) \, \bar{\psi}(\mu)] = 0. \tag{33}$$

Equation (33) is the boundary condition for the diffusion equation given by

Eq. (14). We now algebraically manipulate this result to put it into a more conventional and explicit form.

We rewrite the expression for $\psi(\mathbf{\Omega})$, Eq. (11), making use of Eq. (12), as

$$\psi(\mathbf{\Omega})=\frac{1}{4\pi}\left[\frac{R\coth R-(\mathbf{\Omega}\cdot\nabla E)/(\sigma\omega E)}{(R\coth R)^2-(\mathbf{\Omega}\cdot\mathbf{R})^2}\right], \tag{34}$$

and hence, since ∇E is predominantly perpendicular to the surface,

$$\bar{\psi}(\mu)=\frac{1}{2R^2}\left[\frac{R\coth R-(\mu\,\partial E/\partial z)/(\sigma\omega E)}{\coth^2 R-\mu^2}\right]. \tag{35}$$

Thus we have, since the z axis points along $-\mathbf{n}$,

$$cE\bar{\psi}(\mu)=\frac{1}{2R^2}\left[\frac{(R\coth R)\,E+\mu(\mathbf{n}\cdot\nabla E)/(\sigma\omega)}{\coth^2 R-\mu^2}\right]. \tag{36}$$

Using Eq. (36) in Eq. (33) and grouping terms in a convenient way, we arrive at

$$F_{\text{inc}}=c\gamma[\alpha E+\beta(\mathbf{n}\cdot\nabla E)/\sigma\omega], \tag{37}$$

as the boundary condition for the diffusion equation at each surface point $\mathbf{r}_s$ and time t. Here F_{inc} is the incident radiative flux at $\mathbf{r}_s$ and t, given by

$$F_{\text{inc}}=\int_0^1 d\mu\,\mu\bar{\Gamma}(\mu)=\int_{\mathbf{n}\cdot\mathbf{\Omega}<0} d\mathbf{\Omega}\,|\mathbf{n}\cdot\mathbf{\Omega}|\,\Gamma(\mathbf{\Omega}), \tag{38}$$

and the coefficients α, β, and γ in Eq. (37) are defined as

$$\alpha=\frac{w\coth R}{8(1-w)^{1/2}R}\int_0^1 d\mu\,\frac{\mu H(\mu)}{(v_o-\mu)(\coth^2 R-\mu^2)}, \tag{39}$$

$$\beta=\frac{w}{8(1-w)^{1/2}R^2}\int_0^1 d\mu\,\frac{\mu^2 H(\mu)}{(v_o-\mu)(\coth^2 R-\mu^2)}, \tag{40}$$

$$\gamma=\frac{4(1-w)^{1/2}}{w}\,\frac{\int_0^1 d\mu\,\mu\bar{\Gamma}(\mu)}{\int_0^1 d\mu(\mu H(\mu)\,\bar{\Gamma}(\mu)/(v_o-\mu))}. \tag{41}$$

The function $H(\mu)$ occurring in the above expressions for α, β, and γ is Chandrasekhar's H-function [9] which is related in a simple way to the weight function $W(\mu)$ [8].

We note that Eq. (37) is a mixed (Robbin) boundary condition, which is quite common for a diffusion equation. However, the coefficients α, β, and ω in this boundary condition depend in a complex nonlinear way upon the solution E. Hence this boundary condition, just as the diffusion equation itself, is nonlinear. We also note that α and β are functions of two variables, namely the flux limiting parameter R as defined by Eq. (12), and the single scatter albedo w as defined by Eq. (31). The coefficient γ is a function of the single variable w, but is also a functional of the incoming distribution $\bar{\Gamma}(\mu)$.

It is instructive to consider several limiting cases of the general boundary condition given by Eq. (37). We first consider the coefficient γ in this equation, which we write

$$\gamma = \gamma[w, \bar{\Gamma}(\mu)]. \tag{42}$$

For $\bar{\Gamma}(\mu) = K$, a constant independent of μ (isotropic incidence), it is easily shown that

$$\gamma[w, K] = 1, \tag{43}$$

i.e., the functional γ is unity for all values of w. For pure scattering ($w = 1$) and pure absorption ($w = 0$), we find

$$\gamma[1, \bar{\Gamma}(\mu)] = \frac{4}{3^{1/2}} \frac{\int_0^1 d\mu\, \mu \bar{\Gamma}(\mu)}{\int_0^1 d\mu\, \mu H(\mu)\, \bar{\Gamma}(\mu)}. \tag{44}$$

$$\gamma[0, \bar{\Gamma}(\mu)] = \frac{1}{\bar{\Gamma}(1)} \int_0^1 d\mu\, \mu \bar{\Gamma}(\mu). \tag{45}$$

We tabulate results from Eqs. (44) and (45) in Table I for the special case $\bar{\Gamma}(\mu) = \mu^n$, $n = -1$, 0, 1, using the numerical values for the moments of the H-function given by Chandrasekhar [9]. From this table we conclude that $\gamma[w, \bar{\Gamma}(\mu)]$ depends in a moderate way (neither exceptionally strongly or weakly) upon both the single scatter albedo w, and the incident angular distribution $\bar{\Gamma}(\mu)$.

We next consider the coefficients α and β, which we write

$$\alpha = \alpha(w, R); \qquad \beta = \beta(w, R). \tag{46}$$

We find in the purely scattering ($w = 1$) case

$$\alpha(1, R) = \frac{3^{1/2} \coth R}{8R} \int_0^1 d\mu \frac{\mu H(\mu)}{\coth^2 R - \mu^2}, \tag{47}$$

$$\beta(1, R) = \frac{3^{1/2}}{8R^2} \int_0^1 d\mu \frac{\mu^2 H(\mu)}{\coth^2 R - \mu^2}, \tag{48}$$

and in the purely absorbing ($w = 0$) case

$$\alpha(0, R) = \frac{\sinh(2R)}{8R}, \tag{49}$$

$$\beta(0, R) = \frac{\sinh^2 R}{4R^2}. \tag{50}$$

TABLE I

The Coefficient $\gamma[w, \mu^n]$

n	$w=1$ (Pure scattering)	$w=0$ (Pure absorption)
−1	1.1547	2.0000
0	1.0000	1.0000
1	0.9384	0.6667

Equations (47) through (50) lead to the following limits for small and large R:

$$\alpha(1,0)=1/4; \qquad \beta(1,0)=0.710446/4, \tag{51}$$

$$\alpha(0,0)=\beta(0,0)=1/4, \tag{52}$$

$$\alpha(1,\infty)=0.6296; \qquad \beta(1,R)\xrightarrow[R\to\infty]{} 0.6296/R. \tag{53}$$

For arbitrary w and small R, we find

$$\alpha(w,0)=1/4; \qquad \beta(w,0)=[2\nu_0-w(1-w)^{-1/2}H_1]/8, \tag{54}$$

where H_1 is the first moment of the H-function, tabulated as a function of w by Chandrasekhar [9]. Table II gives $\beta(w,0)$ for various values of w. We note from this table a relatively weak dependence of $\beta(w,0)$ upon w. For arbitrary w and large R, we find

$$\alpha(w,\infty)=\frac{wH(1)}{8(1-w)^{1/2}(\nu_o-1)}, \tag{55}$$

$$\beta(w,R)\xrightarrow[R\to\infty]{}\frac{wH(1)}{8(1-w)^{1/2}(\nu_o-1)\,R}, \tag{56}$$

where $H(1)$ is the H-function evaluated at $\mu=1$.

It is particularly instructive to examine the case of a purely scattering ($w=1$) source-free ($B=0$) medium with weak gradients ($R=0$) and an isotropic incident intensity [$\bar{\Gamma}(\mu)=\text{constant}$]. These conditions imply $\omega=1$, and the appropriate limits for α, β, and γ are given by Eqs. (43) and (51). Using these in the boundary condition given by Eq. (37), we find

$$F_{\text{inc}}=\frac{c}{4}\,[E+(0.710446)(\mathbf{n}\cdot\nabla E)/\sigma]. \tag{57}$$

We see that the cE term on the right-hand side of Eq. (57) is multiplied by $\frac{1}{4}$, which is required if the diffusion description is to give the exact transport solution for a purely scattering, source-free halfspace with an isotropic incident flux. Additionally,

TABLE II

The Coefficient $\beta(w, 0)$

w	$\beta(w, 0)$
0.	0.2500
0.1	0.2432
0.2	0.2351
0.3	0.2259
0.4	0.2165
0.5	0.2078
0.6	0.2000
0.7	0.1933
0.8	0.1874
0.9	0.1822
0.95	0.1798
1.00	0.1776

we see that the $\mathbf{n} \cdot \nabla E$ term in Eq. (57) is multiplied by $(0.710446)/\sigma$, the linear extrapolation distance for the purely scattering Milne problem [8]. This is an intuitively appealing result.

Finally, we consider the case of a source-free, spatially independent halfspace with a single discrete (asymptotic) mode extant. This is the case treated earlier by Pomraning [6]. Here we have

$$w = \omega = \text{constant}, \tag{58}$$

and R, ν_o, and ω are related according to

$$R\nu_o\omega = 1; \qquad 1 = \omega\nu_o \tanh^{-1}(1/\nu_o), \tag{59}$$

$$\nu_o = \coth R; \qquad \omega = (\tanh R)/R. \tag{60}$$

Omitting the straightforward but considerable algebraic detail, we find in this case that

$$\beta/(\alpha\omega) = d = \text{linear extrapolation distance}, \tag{61}$$

$$\alpha\gamma = \nu_o \left[\frac{(1-\omega)[1-(1-\omega)\,\nu_o^2]}{2(\nu_o^2-1)(\nu_o^2-d^2)} \right]^{1/2} \frac{\int_0^1 d\mu\, \mu\Gamma(\mu)}{\int_0^1 d\mu (\mu\Gamma(\mu)/(\nu_o^2-\mu^2)\, X(-\mu))}, \tag{62}$$

where the function $X(-\mu)$ is related to $H(\mu)$ by [8]

$$H(\mu)(1-w)^{1/2}(\nu_o+\mu)\, X(-\mu) = 1. \tag{63}$$

The function $X(-\mu)$ is discussed and tabulated for various values of w by Case and Zweifel [8]. The results given by Eqs. (61) and (62) agree with those given earlier by Pomraning [6], who considered this case alone.

In summary, Eq. (37) represents the boundary condition for the flux-limited diffusion equation given by Eq. (14) in the general case. The material beginning with Eq. (42) deals with various limiting and special cases of this general result. Equations (14), (22), and (37), then, constitute the complete flux-limited diffusion approximation to the equation of transfer, consisting of a diffusion equation given by Eq. (14), an initial condition given by Eq. (22), and a mixed (Robbin) boundary condition given by Eq. (37).

5. Concluding Remarks

Although the equation of transfer is linear in the specific intensity of radiation, the flux-limited diffusion approximation is nonlinear in the radiative energy density, through the parameters R and ω as defined by Eqs. (12) and (13). In particular, the boundary condition given by Eq. (37) is nonlinear because, in addition to the explicit ω appearing in this equation, the parameters α and β depend upon R. These nonlinearities preclude any large class of analytic solutions, which then implies the need for a numerical solution method. A natural question which arises in this context is how one might treat these nonlinearities in a numerical setting. Although a detailed discussion of this question is well beyond the scope of this paper, a few words might be in order.

To date, these nonlinearities have generally been handled by treating them explicitly in a time dependent setting. That is, the diffusion coefficient $\lambda/\sigma\omega$ in Eq. (14), and any nonlinearities in the boundary condition such as in α, β, and ω in Eq. (37), are lagged in time, using results from the solution at the old time step. An obvious improvement to this explicit treatment (but obviously more expensive per time step in computing time) is to iterate the nonlinearities to achieve a time implicit treatment. Another possibility, as yet untried, for treating the nonlinearities in both the differential equation and the boundary condition is a complete Newton–Raphson linearization. The best strategy among these three, and other, possibilities is unknown at this time, and may well be problem dependent. The numerical analysis dealing with flux-limited diffusion theory is a subject needing and deserving a great deal of attention.

Acknowledgment

This work was partially supported by the National Science Foundation.

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